ON THE SOLVABILITY OF PROBLEMS OF THE THEORY OF ELASTICITY OF CONTACT TYPE

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Theorems of existence and uniqueness are proved for problems of the theory of elasticity of contact type. The problem of the Cosserat-Mikhlin spectrum is also investigated.

1. Problems of the theory of elasticity will be termed problems of contact type when boundary conditions of the following form (*)

$$a_{ik}^{(q)}\sigma_{kj}l_j + b_{ik}^{(q)}u_k = N_i^{(q)}, \qquad \langle q = 1, 2, \ldots, N\rangle \tag{1.1}$$

are given for each of the parts $\Sigma^{(q)}$ of the boundary Σ of the body under consideration, which occupies the volume V. Here σ_{kj} are the components of the stress tensor, u_k are the components of the displacement vector, N_i are the contact tractions, and l_i are the direction cosines of the normal to the surface. The elements of the matrices $A^{(q)}$ are dimensionless, while those of the matrices $B^{(q)}$ have the dimensions of a modulus of elasticity divided by length. We shall assume at the beginning that the matrices $A^{(q)}$ and $B^{(q)}$ are diagonal with nonnegative elements

$$A^{(q)} = \begin{Vmatrix} a_1^{(q)} & 0 & 0 \\ 0 & a_2^{(q)} & 0 \\ 0 & 0 & a_3^{(q)} \end{Vmatrix}, \qquad B^{(q)} = \begin{Vmatrix} b_1^{(q)} & 0 & 0 \\ 0 & b_2^{(q)} & 0 \\ 0 & 0 & b_3^{(q)} \end{Vmatrix}$$

If $a_k^{(q)} = 0$, then necessarily in this case $b_k^{(q)} \neq 0$ and on the part of the surface $\Sigma^{(q)}$ the displacement in the k direction is specified

$$u_k = N_k^{(q)} / b_k^{(q)} \qquad \langle k = 1, 2, 3\rangle$$

If $b_k^{(q)} = 0$, then $a_k^{(q)}$ is necessarily nonzero and the traction in the k direction is specified on the part of the surface $\Sigma^{(q)}$

$$\sigma_{ki}l_i = N_k^{(q)} / a_k^{(q)} \qquad \langle k = 1, 2, 3\rangle$$

In particular, so-called mixed boundary conditions are obtained if q takes on the values 1 and 2; the surface tractions S_i° are specified on $\Sigma^{(1)}$ and the displacements u_i° are specified on $\Sigma^{(2)}$. For this case

$$a_{ij}^{(1)} = \delta_{ij}, \qquad b_{ij}^{(1)} = 0, \qquad N_i^{(1)} = S_i^\circ$$
$$a_{ij}^{(2)} = 0, \qquad b_{ij}^{(2)} = C\delta_{ij}, \qquad N_i^{(2)} = Cu_i^\circ$$

where C is some quantity having the same dimensions as the elements of the matrices $B^{(q)}$

2. The problem of the theory of elasticity consists of integrating the equations of motion

$$\sigma_{ij,j} + \rho F_i = \rho u_i^{\cdot\cdot} \tag{2.1}$$

so that the boundary conditions (1.1) and some initial conditions are satisfied. Here the

*) Summation from 1 to 3 on repeated subscripts is to be understood except in cases where the values of the scripts are enclosed in angle brackets.

stress tensor is expressed in terms of the strain tensor by Hooke's law for an isotropic material

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij}$$

and the tensor of small strains ε_{ij} is expressed in terms of the displacements u_i by the relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \theta = \varepsilon_{kk} = u_{k,k}$$

The solution of the problem which has been posed is unique. Indeed, it follows from the equations of motion (2.1) that for the difference between two solutions $u_i = u_i^{(2)} - u_i^{(1)}$

$$\int\limits_V [\sigma_{ij}\varepsilon_{ij}^{\cdot} + \rho u_i^{\cdot\cdot} u_i^{\cdot}]\, dV = \int\limits_\Sigma \sigma_{ij} l_j u_i^{\cdot}\, d\Sigma \tag{2.2}$$

We split the integral on the right-hand side of (2.2) into the sum of integrals over the surfaces $\Sigma^{(q)}$. On each of these surafces we define a diagonal matrix $\Gamma^{(q)}$ according to the following rule. An element $\gamma_k^{(q)}$ of the matrix $\Gamma^{(q)}$ is zero if at least one of the corresponding elements of the matrices $A^{(q)}$ and $B^{(q)}$ ($a_k^{(q)}$ or $b_k^{(q)}$) is zero, and

$$\gamma_k^{(q)} = b_k^{(q)} / a_k^{(q)}$$

if the elements $a_k^{(q)}$ and $b_k^{(q)}$ are both nonzero. Then, as is clear from the homogeneus boundary conditions (1.1) which are satisfied by the difference solution,

$$\int\limits_\Sigma \sigma_{ij} l_j u_i^{\cdot}\, d\Sigma = -\frac{1}{2}\sum_{q=1}^{N} \int\limits_{\Sigma^{(q)}} \gamma_{ij}^{(q)} \frac{\partial}{\partial t}(u_i u_j)\, d\Sigma$$

Thus, the relation (2.2) can be written in the following form:

$$\frac{\partial}{\partial t}\left\{\int\limits_V \left[\left(\lambda + \frac{2}{3}\mu\right)\theta^2 + 2\mu e_{ij} e_{ij}\right] dV + \frac{1}{2}\rho u_i^{\cdot} u_i^{\cdot}\right\} + \sum_{q=1}^{N} \int\limits_{\Sigma^{(q)}} \gamma_{ij}^{(q)} u_i u_j d\Sigma = 0 \tag{2.3}$$

The uniqueness of the solution of the problem which has been formulated then follows by virtue of the fact that each term of (2.3) is positive and that the initial conditions are homogeneous, provided that Poisson's ratio ν is in the following range:

$$-1 \leqslant \nu < {}^1/_2 \tag{2.4}$$

(It is obvious that uniqueness still holds in the case where $a_k^{(q)}$ and $b_k^{(q)}$ are both strictly negative).

The uniqueness of the quasi-static problem, for which the right-hand sides of Eqs. (2.1) are taken as zero, follows from what has been proved as a special case. In this case the equations of equilibrium can be written in the following form:

$$u_{i,jj} + \varkappa u_{j,ij} = f_i \tag{2.5}$$

and the boundary conditions (1.1) on the part $\Sigma^{(q)}$ of the surface Σ in the form

$$a_{ik}^{(q)}[(\varkappa - 1) l_k u_{j,j} + l_j(u_{k,j} + u_{j,k})] + b_{ik}^{(q)} u_k = n_i^{(q)} \tag{2.6}$$

$$\varkappa \equiv \frac{1}{1-2\nu}, \quad f_i \equiv -\frac{\rho F_i}{\mu}, \quad n_i^{(q)} \equiv \frac{N_i^{(q)}}{\mu}$$

3. Papers [1, 2] are devoted to the investigation of the Cosserat spectrum, i.e. to the problem of the eigenvalues $\varkappa$ and eigenfunctions of the first two boundary value problems of the statical theory of elasticity. We shall make use of a theorem of Mikhlin [2] to study this question in problems of contact type, Eqs. (2.5) and (2.6).

Let the operational equation

$$(\mathbf{R} - \tau\mathbf{T})\,\mathbf{u} = 0 \tag{3.1}$$

be given and satisfy the following conditions:

R is a positive definite differential operator;

T is a differential operator which is symmetric for the given boundary conditions;

the operator $\mathbf{R} - \tau\mathbf{T}$ is assumed to be elliptic for all values of τ except the discrete set M_1;

for the given boundary conditions the complementing condition [3] is satisfied for all τ except the discrete set M_2.

Then the following statements are correct:

1. The spectrum of Eq. (3.1) can have as points of accumulation only points of the set M_1 or of the set M_2.
2. The system of eigenfunctions of Eq. (3.1) is complete in H, the energy space of the operator **R**.
3. The eigenvalues are real.
4. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the metric of the space H.

Let us consider the homogeneous system of equations

$$\Delta^* u_i \equiv u_{i,jj} + \varkappa u_{j,ij} = 0 \tag{3.2}$$

and the homogeneous boundary conditions of contact type

$$a_{ik}^{(q)} [(\varkappa - 1) l_k u_{j,j} + l_j (u_{k,j} + u_{j,k})] + b_{ik}^{(q)} u_k = 0 \tag{3.3}$$

By the operator **R**, we shall mean the operator

$$\mathbf{R} u_i \equiv - u_{i,jj} - u_{j,ij} \tag{3.4}$$

the boundary conditions corresponding to the operator **r** acting on Σ

$$-\mathbf{r} u_i \equiv a_{ik}^{(q)} l_j (u_{k,j} + u_{j,k}) + b_{ik}^{(q)} u_k \tag{3.5}$$

The operator **T** will be

$$\mathbf{T} u_i \equiv u_{j,ij} \tag{3.6}$$

with the operator **t** corresponding to the boundary conditions on each of the $\Sigma^{(q)}$

$$\mathbf{t} u_i \equiv a_{ik}^{(q)} l_k u_{j,j} \tag{3.7}$$

The homogeneous boundary value problem (3.2) and (3.3) then corresponds to the operational equation (3.1) with the boundary conditions

$$(\mathbf{r} - \tau\mathbf{t}) u_i = 0 \qquad \left(\tau = \varkappa - 1 = 2\frac{\nu}{1 - 2\nu}\right)$$

We use Betti's third formula [4]

$$\int_V (u_i \Delta^* v_i - v_i \Delta^* u_i)\, dV = \int_\Sigma [u_i P_i(v) - v_i P_i(u)]\, d\Sigma \tag{3.8}$$

$$P_i(u) \equiv (\varkappa - 1) l_i u_{j,j} + (u_{i,j} + u_{j,i}) l_j$$

to prove the positive definiteness of the operator **R**.

Breaking up the right-hand side of (3.8) into a sum of integrals on the $\Sigma^{(q)}$ and forming the matrix $\Gamma^{(q)}$ as before, we have

$$\int_V (u_i \Delta^* v_i - v_i \Delta^* u_i)\, dV = \sum_{q=1}^{N} \int_{\Sigma^{(q)}} \gamma_{ij}^{(q)} (v_i u_j - v_j u_i)\, d\Sigma \tag{3.9}$$

The symmetry of the operator **R** (3.4) for the boundary conditions (3.5) follows from (3.9) with $\varkappa = 1$. Further, using Betti's second formula [4]

$$\int_V u_i \Delta^* u_i dV = \int_\Sigma u_i P_i(u)\, d\Sigma - \int_V E(u, u) \cdot dV \qquad (3.10)$$

The quadratic form

$$E(u, u) \equiv (\varkappa - 1)\, \theta^2 + u_{i,j} u_{i,j} + u_{i,j} u_{j,i}$$

is positive definite for $\varkappa = 1$ (i.e. for $\nu = 0$) by virtue of the uniqueness theorem which has been proved. Since the surface integral on the right-hand side of (3.10)

$$\int_\Sigma u_i P_i(u)\, d\Sigma = -\sum_{q=1}^{N} \int_{\Sigma^{(q)}} \gamma_{ij}^{(q)} u_i u_j d\Sigma \qquad (3.11)$$

is negative and equals zero only for $u_i \equiv 0$, the operator $\mathbf{R}$ obtained from Δ^* for $\varkappa = 1$ is positive definite.

Moreover, the operator $\mathbf{T}$, Eq. (3.6), is symmetric under the boundary conditions (3.7)

$$\int_V (u_{j,ij} v_i - v_{j,ij} u_i)\, dV = \int_\Sigma (u_{j,j} v_i - v_{j,j} u_i)\, l_i d\Sigma \equiv 0$$

The differential equation (3.2) is elliptic in the sense of I. G. Petrovskii for all real $\varkappa$ except $\varkappa = -1$ and $\varkappa = \infty$. Calculation shows that the "complimenting condition" of [3] is satisfied for all $\varkappa$ except $\varkappa = -2, -1, 0, \infty$. Here it can be considered that the elements of the matrices A and B are sufficiently smooth functions of the coordinates on Σ. The premises of Mikhlin's theorem are satisfied, and therefore the statements 1–4 hold. The operator of the problem (3.2), (3.3) is normally solvable and has a finite index [5] for all $\varkappa$ except $\varkappa = -2, -1, 0, \infty$. Other eigenvalues of the problem (3.2), (3.3) have finite multiplicity.

The Agmon-Douglis-Nirenberg determinants which occur in the "complementing boundary conditions" are zero only for those parts $\Sigma^{(q)}$ of the surface Σ on which only displacements or only tractions are specified. Therefore, the points $\varkappa = -\infty$, $\varkappa = -1$ are eigenvalues of infinite multiplicity [2].

Knowing the eigenfunctions $u_i^{(n)}$ of the problem (3.2), (3.3) (we shall assume that they are orthonormal in the energy space H of the operator $\mathbf{R}$), we can easily find the solution of the inhomogeneous problem (2.5), (2.6)

$$u_i = \sum_{n=1}^{\infty} \frac{(\varkappa_n - 1)\, f_n}{\varkappa_n - \varkappa}\, u_i^{(n)}, \qquad f_n \equiv (f_i, u_i^{(n)})$$

In particular, the theorem of existence of the solution of the problem of the theory of elasticity with boundary conditions of contact type follow from this, since in the range (2.4) there are no points of the spectrum of the problem (3.2), (3.3), by virtue of the theorem of uniqueness which has been proved.

4. Now let the matrices A and B which occur in the relations (1.1) not be necessarily diagonal. We shall assume that these matrices are singular if, and only if, they have only zeros in the kth row or the kth column. We now construct the matrix

$$\Gamma = \mathrm{A}'\mathrm{B} \qquad (4.1)$$

If the matrix A is nonsingular, then $\mathrm{A}' = \mathrm{A}^{-1}$, where A^{-1} is the inverse of A. If A is singular, then the following three cases can arise:

1) A is singular (i.e. there are only zeros in the kth row or kth column), but the matrix A_2 formed by striking out the kth row and kth column of A is nonsingular.

2) Both A and A_2 are singular (there are only zeros in the l th row or l th column), but A_1 formed from A_2 by striking out the l th row and l th column is nonsingular.

3) All three matrices A, A_2 and A_1 are singular.

In Case (1) we place zeros in the k th row and kth column of the matrix A', and the remaining elements are those of A_2^{-1}. In Case (2) the elements of the k th and l th rows and columns of the matrix A' are set equal to zero and in the remaining place we put A_1^{-1}. Finally, in Case (3) $A' \equiv 0$.

It is easy to see now that if the matrix Γ (4.1) is positive definite, then all the statements proved above for diagonal matrices A and B remain valid, since the quadratic forms $\gamma_{ij}^{(q)}u_i u_j$ entering into the relations (2.3) and (3.11) are also positive definite in the present case.

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ON THE ANALYSIS OF SHELLS WITH HOLES

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The possibility of application of Neumann's procedure to the investigation of shells with holes is examined.

The transfer of Neumann's method to shells is connected with two difficulties. First of all, the application of Kirchhoff's stresses does not reduce the problem directly to well-studied integral equations. Therefore, in this paper initially the investigation is related to the principal vector and moment. The latter circumstance, however, leads to singular integral equations which also have fixed singularities. However, the specific form of the resulting equations allows the establishment of Fredholm's alternative for these equations within the required limits. After the proof of Fredholm's alternative the convergence of Neumann's method is proven. The results of Fredholm present the possibility to establish convergence of Kirchhoff's stresses for sufficiently smooth contours of holes and load.

We shall examine shells with holes for which Neumann's method can be realized on